University of Bahrain

SERIAL

College of Information Technology
Department of Computer Science

ITCS253 Discrete Structures II

First Semester 2015/2016

First Exam – 75 Minutes

*** Key Solution ***

STUDENT NAME	**** **Key Solution** ****
STUDENT#	**** **Key Solution** ****
SECTION	**** **Key Solution** ****

- ▶ This exam contains **5 pages** (including this cover page) and **5 questions**. Check to see if any pages are missing.
- ▶ You are **allowed** to use Calculators.
- ▶ You are **not allowed** to use books, notes, or mobiles.
- ▶ Please write **one answer**. In case of writing multiple answers, mistakes in any answer will be counted.

Question	Points	Score
1	7	
2	5	
3	4	
4	7	
5	7	
Total:	30	

Instructor: Dr. Ali Alsaffar Sections# 1 & 2

Answer all questions

(1) Answer the following questions.

(a) [**1 point**] Let $f : \mathbf{Z} \to \mathbf{Z}$ defined by $f(n) = n + \sqrt{n}$. Why f is not a function.

> **Solution:** Because for $x = -1 \in \mathbf{Z} \implies f(1) = -1 + \sqrt{-1} \notin \mathbf{Z}$.

(b) [**2 points**] Let $A = \{1, 2, 3\}, B = \{0, -1\}, C = \{0, -1, 4\}$, and $D = \{6, 7\}$. Suppose $f : A \to B$ and $g : C \to D$ be two functions. Do $(f \circ g)$ and $(g \circ f)$ exists. Justify your answer.

> **Solution:** For $f : A \to B$ and $g : C \to D$. Since $D \nsubseteq A$, then, $f \circ g$ does not exist.
> For $f : A \to B$ and $g : C \to D$. Since $B \subseteq A$, then, $f \circ g$ exists.

(c) [**2 points**] Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x) = x + \dfrac{x}{|x|}$. Suppose $f(x) = 0$, what is the value of x (if it exists.)

> **Solution:** $x + \dfrac{x}{|x|} = 0 \implies \dfrac{x|x| + x}{|x|} = 0 \implies x|x| + x = 0 \implies x|x| = -x \implies |x| = -1$. Hence, x does not exists because $|x| \geq 0 \neq -1$.

(d) [**2 points**] Let $f, g, h : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = x + 2, \qquad g(x) = \frac{1}{x^2 + 1}, \qquad h(x) = 3$$

Find $f^{-1} \circ g \circ h(x)$.

> **Solution:** Let $f^{-1} = y$. Then, $f(y) = x$ and $y + 2 = x$. Solve for x we get $y = x - 2$.
> Hence, $f^{-1} = x - 2$ and
> $f^{-1} \circ g \circ h(x) = f^{-1} \circ g(h(x)) = f^{-1} \circ g(3) = f^{-1}(g(3)) = f^{-1}(1/10) = -1.9$

(2) [**5 points**] Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x) = 3x^2 - 2x + 1$.
Find the range and prove that f is not one-to-one and not onto.

Solution:

1. **Range:**
   Let $y = 3x^2 - 2x + 1$, then $y' = 0$ is $6x - 2 = 0$ and $x = 1/3$.
   $\therefore f(1/3) = 3(1/3)^2 - 2(1/3) + 1 = 2/3 = .6667$.
   Since the coefficient of x^2 is positive, then the range is $\{y \in \mathbf{R} \mid y \geq 2/3\}$.

2. one-to-one
   For any $x, y \in \mathbf{R}$, let $f(x) = f(y)$, then

$$3x^2 - 2x + 1 = 3y^2 - 2y + 1 \implies$$
$$3(x^2 - y^2) = 2(x - y) \implies$$
$$(x - y)(x + y) = (2/3)(x - y) \implies$$

   Case #1: if $x = y$, then LHS=RHS and we are done.
   Case #2: if $x \neq y$, then $\cancel{(x - y)}(x + y) = (2/3)\cancel{(x - y)}$ and $x + y = 2/3$ which has many solutions. For example, $x = 1$ and $y = -1/3$, hence, f is not one-to-one.

3. Since *range* $f \neq$ *co-domain* f, then f is not onto.

(3) [**4 points**] Suppose G is a group. If $a * b^{-1} = (a^{-1} * b)^{-1}$, show that G is an abelian for any $a, b \in G$.

Solution: *Proof.* Let $a * b^{-1} = (a^{-1} * b)^{-1}$.

$$\therefore a * b^{-1} = b^{-1} * a \qquad \text{because } (a^{-1} * b)^{-1} = b^{-1} * a$$
$$b * a * \underbrace{b^{-1} * b}_{e} = \underbrace{b * b^{-1}}_{e} * a * b \qquad \text{left and right multiply by } b$$
$$b * a = a * b$$

(4) Given a binary operation * on a set $G = \{a, b, c, d\}$, where * is described by the table below.

$*$	a	b	c	d
a	c	d	a	b
b		a	b	c
c	a		c	d
d	b	c	d	

(a) **[3 points]** Assume that * is associative. Find the missing entries in the table.

> **Solution:** $b * a = (d * a) * a = d * (a * a) = d * c = d$
> $c * b = (d * b) * b = d * (b * b) = d * a = b$
> $d * d = (a * b) * d = a * (b * d) = a * c = a.$

(b) **[3 points]** Is $(G, *)$ a group. Explain and give reasons to your answer.

> **Solution:**
>
> 1. *Closure.* Since every entry in the table belongs to $\{a, b, c, d\}$. Then, G is closed.
> 2. *Associative.* It is given that * is associative.
> 3. *Identity Element e.* From the table $e = c$.
> 4. *Inverse of each element.* $a^{-1} = a,\ b^{-1} = d,\ c^{-1} = c,\ d^{-1} = b.$

(c) **[1 point]** Is $(G, *)$ an abelian group? Justify your answer

> **Solution:** Since the entries in the table are symmetric with respect to the main diagonal, then * is commutative and hence $(G, *)$ is abelian.

(5) (a) **[2 points]** Let $n = 4^k$, for $n > 0$ and $k \geq 0$. Show that $k = \log_2 \sqrt{n}$.

Solution: Let $n = 4^k$. Take $\log_2$ both sides we get $\log_2 n = \log_2 4^k$.
$\therefore \log_2 n = \log_2 (2^2)^k = \log_2 2^{2k} = 2k \log_2 2 = 2k$.
Then, $k = (1/2) \log_2 n = \log_2 n^{1/2} = \log_2 \sqrt{n}$.
$\therefore k = \log_2 \sqrt{n}$

(b) **[5 points]** Use the *Back Substitution* method to solve the following recurrence relation.

$$T(1) = 7, \quad T(n) = T(n/3) + 5$$

Solution:

$$\begin{aligned}
T(n) &= T(n/3) + 5 \\
&= T(n/3^2) + 5 + 5 \\
&= T(n/3^3) + 5 + 5 + 5 \\
&\vdots \\
&= T(n/3^k) + 5k \qquad \text{(at step } k\text{)} \\
&\vdots \\
&\text{until } T(n/3^k) = T(1) \implies n = 3^k \text{ and } k = \log_3 n \\
&= T(1) + 5 \log_3 n
\end{aligned}$$

$\therefore T(n) = 7 + 5 \log_3 n$